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Cycles in k-traceable oriented graphs

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ABSTRACT

A digraph of order at least k is termed *k-traceable* if each of its subdigraphs of order k is traceable. It turns out that several properties of tournaments—i.e., the 2-traceable oriented graphs—extend to k-traceable oriented graphs for small values of k. For instance, the authors together with O. Oellermann have recently shown that for $k = 2, 3, 4, 5, 6$, all k-traceable oriented graphs are traceable. Moon [J.W. Moon, On subtournaments of a tournament, Canad. Math. Bull. 9(3) (1966) 297–301] observed that every nontrivial strong tournament T is vertex-pancyclic—i.e., through each vertex there is a cycle of every length from 3 up to the order of T. The present paper reports results pertaining to various cycle properties of strong k-traceable oriented graphs and explores the extent to which pancyclicity is retained by strong k-traceable oriented graphs.

For each $k \geq 2$ there are infinitely many k-traceable oriented graphs—e.g. tournaments. However, we establish an upper bound (linear in k) on the order of k-traceable oriented graphs having a strong component with girth greater than 3. As an application of our findings, we show that the Path Partition Conjecture holds for 1-deficient oriented graphs having a strong component with girth at least 6. (A digraph is 1-deficient if its order is exactly one more than the order of its longest paths.)

1. Introduction

Let D be a finite digraph with vertex set $V(D)$ and arc set $A(D)$. The number of vertices of D is called its *order* and is denoted by $n(D)$. The number of arcs of D is called its *size*. For any nonempty set $X \subseteq V(D)$, $\langle X \rangle$ denotes the subdigraph of D induced by X.

If v is a vertex in a digraph D, we denote the sets of *out-neighbours* and *in-neighbours* of v by $N^+(v)$ and $N^-(v)$ and the cardinalities of these sets by $d^+(v)$ and $d^-(v)$, respectively. The *degree* of v in D is defined as $d(v) = d^+(v) + d^-(v)$ and the *minimum degree* of D is $\delta(D) = \min_{v \in V(D)} d(v)$. A digraph D is *t-regular* if $d^+(v) = d^-(v) = t$ for all $v \in V(D)$. The set of nonneighbours of v in D (including v) is denoted by $N^0[v]$; i.e., $N^0[v] = V(D) - N(v)$.

If X is a subdigraph of D or a subset of $V(D)$, we denote the set of neighbours (in-neighbours, out-neighbours) of v in X by $N_X(v)$ ($N_X^-(v)$, $N_X^+(v)$, respectively). For a subdigraph S, $N(S) = \cup_{v \in V(S)} N(v)$.

A digraph D is *strong* if for every two distinct vertices $u, v \in V(D)$ there is a $u - v$ path in D. A maximal strong subdigraph of a digraph D is called a *strong component* of D. We say that a digraph is *trivial* if it has only one vertex.

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We denote the directed t-cycle by $\overrightarrow{C}_t$. In a directed cycle $v_1 \dots v_t v_1$ ($t \geq 4$), the arc $v_i v_{i+2}$, for $i = 1, 2, \dots, t$ (taken modulo t) is called a *short forward chord*. Two chords emanating from consecutive vertices are called *consecutive chords*.

A directed cycle (path) in a digraph will simply be called a *cycle* (*path*). If D is a digraph containing a cycle, then the *girth*, $g(D)$ (*circumference*, $c(D)$), of D is the length of a shortest (longest) cycle in D. A cycle of length $g(D)$ is called a *girth cycle* of D. The order of a longest path in D is denoted by $\lambda(D)$. A digraph D is *hamiltonian* if $c(D) = n(D)$, *traceable* if $\lambda(D) = n(D)$, and *p-deficient* if $\lambda(D) = n(D) - p$.

A digraph containing no 3-cycle is said to be *triangle-free*. An *oriented* graph is a digraph without 2-cycles. A *tournament* is an oriented graph whose underlying graph is complete. If D is an oriented graph such that $\langle N^+(v)\rangle$ as well as $\langle N^-(v)\rangle$ are tournaments for each $v \in V(D)$, then D is called a *local tournament*.

A digraph D of order $n \geq 3$ is *t-pancyclic* ($t \leq n$) if D contains a cycle of length r for each $r = t, t+1, \dots, n$. If every vertex in D belongs to an r-cycle for every $r = t, t+1, \dots, n$, then D is *vertex-t-pancyclic*. In particular, if D is $g(D)$-pancyclic or vertex-$g(D)$-pancyclic, we say simply that D is *girth-pancyclic*, respectively *vertex-girth-pancyclic*. A digraph D is called *weakly (vertex-)pancyclic* if it contains cycles of every length from $g(D)$ to $c(D)$ (through each vertex).

For undefined standard terminology we refer the reader to [1].

A digraph D is *k-traceable* if $n(D) \geq k$ and each of its induced subdigraphs of order k is traceable. Obviously, a nontrivial oriented graph is 2-traceable if and only if it is a tournament. Thus for $k \geq 3$, k-traceable oriented graphs are natural generalizations of tournaments.

It is easily seen that 3-traceable oriented graphs are local tournaments. Bang-Jensen et al. [2] showed that local tournaments retain several properties of tournaments.

It is known that various elementary properties of tournaments generalize to k-traceable oriented graphs for small values of k. For example, for $k = 2, 3, 4$, every strong k-traceable oriented graph of order greater than k is hamiltonian, as shown in [12]. Furthermore, it is shown in [11] that for $k = 2, 3, 4, 5, 6$, every k-traceable oriented graph is traceable.

Moon [7] observed that every strong tournament of order at least 3 is vertex-pancyclic. Powers of cycles, described in Section 2, provide us with interesting examples of k-traceable oriented graphs that are vertex-girth-pancyclic. These examples also serve to illustrate the sharpness of some results in Sections 3 and 4. In Section 3, we consider the minimum size of k-traceable oriented graphs of given order. In Section 4, we show that for $k = 2, 3, 4$, all strong k-traceable oriented graphs of order $n \geq k+1$ are vertex-$(k+1)$-pancyclic. This fact does not extend to k-traceable oriented graphs with $k \geq 5$, although girth-pancyclicity is retained by strong k-traceable oriented graphs with girth at least k.

The Traceability Conjecture (TC), treated in [11,12,5], states that for $k \geq 2$ every k-traceable oriented graph of order at least $2k-1$ is traceable. It is shown in [11] that for each $k \geq 2$ there exists an integer $t(k)$ ($< 2k^2$) such that every k-traceable oriented graph of order at least $t(k)$ is traceable. We have not yet succeeded in reducing the upper bound on $t(k)$ to a function that is linear in k when $k \geq 7$. However, in Section 5 we establish an upper bound (linear in k) on the order of all k-traceable oriented graphs (traceable or not) having a strong component X with $g(X) > 3$. In particular, we show that if $k \geq 2$ and D is a k-traceable oriented graph of order greater than $2k-4$, then every strong component of D has girth at most 5.

The Path Partition Conjecture (PPC) states that if D is any digraph and (a, b) any pair of positive integers such that $a + b = \lambda(D)$, then D has a vertex partition (A, B) such that $\lambda(\langle A\rangle) \leq a$ and $\lambda(\langle B\rangle) \leq b$. For information on the PPC and its connection to the TC, see e.g. [10,12,2]. In Section 6, we apply our findings to show that the Path Partition Conjecture holds for 1-deficient oriented graphs with girth greater than 5.

2. Powers of cycles

The tth *power*, $\overrightarrow{C}_n^t$, *of the n-cycle* $\overrightarrow{C}_n = v_1 v_2 \dots v_n v_1$ is the digraph obtained from $\overrightarrow{C}_n$ by adding, for each $i \in \{1, 2, \dots, n\}$, an arc from v_i to every vertex in $\{v_{i+2}, v_{i+3}, \dots, v_{i+t}\}$ (indices modulo n). Fig. 1 illustrates $\overrightarrow{C}_9^3$.

Proposition 2.1. *Let $D = \overrightarrow{C}_n^t$ be the tth power of the cycle $v_1 v_2 \dots v_n v_1$, $t \geq 1$ and $n \geq 2t+1$. Then the following hold:*

1. *D is a strong t-regular oriented graph;*
2. *D is a local tournament;*
3. *D is vertex-girth-pancyclic with girth $\lceil \frac{n}{t} \rceil$;*
4. *D is k-traceable for $n - 2t + 1 \leq k \leq n$.*

Proof. Items 1 and 2 follow directly from the definition of D.

Moreover, D has girth $g = \lceil \frac{n}{t} \rceil$, since $v_1 v_{t+1} v_{2t+1} \dots v_{(\lceil \frac{n}{t} \rceil - 1)t+1} v_1$ is a shortest cycle in D. Now it is easy to see that each vertex of D is on a cycle of every length from g to n. This proves item 3.

To prove item 4, consider a subdigraph H of D of order k, $n - 2t + 1 \leq k \leq n$, and let $S = V(D) - V(H)$. Then $|S| = n - k \leq 2t - 1$ and hence H is connected. Suppose H has two vertices x and y of in-degree 0. Then S must contain the t vertices immediately preceding x on the cycle as well as the t vertices immediately preceding y on the cycle. But, since x and y are nonadjacent, these $2t$ vertices are all distinct and hence we obtain the contradiction that $|S| \geq 2t$. Thus H contains at most one vertex of in-degree 0. Now, since every vertex on the cycle is adjacent to the next t vertices, it is easy to see that H is hamiltonian. $\square$

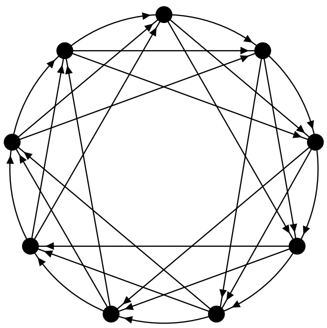

Fig. 1. The 3rd power of the directed 9-cycle—a strong 4-traceable oriented graph with girth 4.

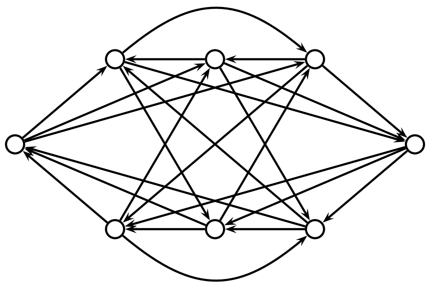

Fig. 2. A 3-traceable oriented graph (which is not a power of a cycle) realizing the bound in Corollary 3.2.

Items 2 and 3 of Proposition 2.1 show that powers of cycles retain certain properties of tournaments. Even so, they provide us with examples of extremely sparse k-traceable oriented graphs, as shown in the next section.

3. Sparse *k*-traceable oriented graphs

The next lemma is proved in [12]; we include the proof for completeness.

Lemma 3.1. *Let D be a k-traceable oriented graph of order n, $2 \leq k \leq n$. Then $\delta(D) \geq n - k + 1$.*

Proof. Suppose, to the contrary, that D has a vertex x with $d(x) \leq n - k$. Then $|N^0[x]| \geq k$. Let H be an induced subdigraph of D such that $n(H) = k$ and $x \in V(H) \subseteq N^0[x]$. Then H is nontraceable, a contradiction. □

Lemma 3.1 provides a lower bound for the size of k-traceable oriented graphs of order n.

Corollary 3.2. *The size of a k-traceable oriented graph of order n is at least $\lceil \frac{n(n-k+1)}{2} \rceil$.*

The seemingly trivial lower bound in Corollary 3.2 is realized by the oriented graph $\overrightarrow{C}_n^{(n-k+1)/2}$ when $n - k + 1$ is even, i.e., when n and k have different parities. Other examples of k-traceable oriented graphs with $d(v) = n - k + 1$ for all $v \in V(D)$ include tournaments of order n (for the case $k = 2$) and the oriented graph depicted in Fig. 2 (for the case $k = 3$).

For the case $k \geq 4$ we do not know of any k-traceable oriented graphs with $d(v) = n - k + 1$ for all $v \in V$, other than powers of cycles. However, we do know that such digraphs would have to be strong:

Proposition 3.3. *Let D be k-traceable oriented graph of order n, $n \geq k \geq 3$. If $d(v) = n - k + 1$ for all $v \in V(D)$, then D is strong.*

Proof. Suppose D is not strong and let its strong components be $D_1, D_2, \ldots, D_h$, $h \geq 2$, such that no arc exists from D_i to D_j if $i > j$. (This is always possible, cf. [1].) Let $x \in V(D)$ be such that x has an out-neighbour y. Suppose $x \in V(D_i)$ with $i \geq 2$. Since $\langle \{y\} \cup N^0[x] \rangle$ has order k, it is traceable and therefore $N^0[x] \subseteq V(D_i) \cup V(D_{i+1}) \cup \cdots \cup V(D_h)$. Let $z \in V(D_{i-1})$. Then $z \in N^-(x)$ and hence $\langle \{z\} \cup N^0[x] \rangle$ is a subdigraph of D of order k that is nontraceable—indeed, in this subdigraph, z is a source, x is a sink, and x is nonadjacent to all vertices of $N^0(x)$. This contradiction shows that $x \in V(D_1)$.

Similarly, if x has an in-neighbour, then $x \in V(D_h)$. It follows that $h = 2$. Moreover, since no vertex in D_1 (respectively, D_2) has an in-neighbour (respectively, out-neighbour), $|V(D_1)| = 1 = |V(D_2)|$, implying the contradiction that $n = 2$. □

The following lemma is proved in [11]; we include the proof for completeness.

Lemma 3.4 *([11]). If D is a k-traceable oriented graph of order n and x, y are two nonadjacent vertices in D, then $|N^+(x) \cup N^+(y)| \geq n-k+1$ and $|N^-(x) \cup N^-(y)| \geq n-k+1$. In particular, if $d^+(x)+d^+(y) = n-k+1$ (respectively, $d^-(x)+d^-(y) = n-k+1$), then $N^+(x) \cap N^+(y) = \emptyset$ (respectively, $N^-(x) \cup N^-(y) = \emptyset$).*

Proof. Suppose, to the contrary that $|N^+(x) \cup N^+(y)| \leq n-k$. Then $|V(D) \setminus (N^+(x) \cup N^+(y))| \geq k$. But every subdigraph of $D - (N^+(x) \cup N^+(y))$ that contains both x and y is nontraceable, since neither x nor y has any out-neighbour in $V(D) \setminus (N^+(x) \cup N^+(y))$. This contradicts our assumption that D is k-traceable.

The proof that $|N^-(x) \cup N^-(y)| \geq n-k+1$ is similar. $\square$

Corollary 3.5. *Let n and k have the same parity and suppose $\{x_1, x_2, x_3\}$ is an independent set in a k-traceable oriented graph of order n. Then at least one of x_1, x_2, x_3 has degree at least $n-k+2$.*

Proof. Suppose, to the contrary, each of x_1, x_2, x_3 has degree at most $n-k+1$. Since $n-k+1$ is odd by our assumption, we may assume $d^+(x_1) \leq \frac{n-k}{2}$. Then Lemma 3.4 implies that $d^+(x_i) \geq \frac{n-k+2}{2}$ for $i = 2, 3$. But then $d^-(x_i) \leq \frac{n-k}{2}$ for $i = 2, 3$, which contradicts Lemma 3.4. $\square$

If n and k have the same parity ($n \geq k \geq 3$), then $D = \overrightarrow{C}_n^{(n-k+2)/2}$ is a k-traceable oriented graph of order n and $d(v) = n-k+2$ for all $v \in V(D)$. The graph has size $\frac{n(n-k+2)}{2}$, which is somewhat bigger than the bound of Corollary 3.2. For $n > k \geq 3$ it is unlikely that any spanning subdigraph of $\overrightarrow{C}_n^{(n-k+2)/2}$ realizes the bound of Corollary 3.2, because Corollary 3.5 shows that not many arcs can be deleted from a k-traceable oriented graph with $d(v) = n-k+2$ for all $v \in V(D)$ without destroying the k-traceability. We note for example that $\overrightarrow{C}_6^2$ *minus two consecutive short forward chords* is a 4-traceable oriented graph of order 6 and size 10. The bound given by Corollary 3.2 in this case is 9. However, deleting any three arcs from $\overrightarrow{C}_6^2$ destroys the 4-traceability.

The path $\overrightarrow{P_3}$ realizes the bound in Corollary 3.2 for the case $n = k = 3$. For $n > k \geq 3$ it remains an open problem to establish a sharp lower bound for the size of k-traceable oriented graphs of order n, with n and k having the same parity.

4. Pancyclicity

For every $k \geq 2$, powers of cycles provide infinitely many examples of strong k-traceable oriented graphs that are vertex-girth-pancyclic. In this section we present vertex-pancyclicity and girth-pancyclicity results for strong k-traceable oriented graphs. First we observe the following.

Lemma 4.1. *If D is a strong k-traceable oriented graph ($k \geq 2$) of order at least 3, then for every $v \in V(D)$ there is a cycle of order at most $k+1$ containing v. In particular, $g(D) \leq k+1$.*

Proof. Since D is strong, v is contained in a cycle $C = vv_2v_3 \ldots v_jv$. If $j \geq k+2$, then v has a neighbour in $\{v_3, v_4, \ldots, v_{k+1}\}$, since $\langle\{v, v_3, v_4, \ldots, v_{k+1}\}\rangle$ is traceable. Therefore, v is contained in a cycle shorter than C. $\square$

It is well known that every strong tournament of order $n \geq 3$ contains a cycle of length n, i.e., every strong 2-traceable oriented graph of order at least 3 is hamiltonian. The following extension of this result is proved in [12].

Theorem 4.2 *([12]). For $k = 2, 3, 4$ every strong k-traceable oriented graph of order at least $k+1$ is hamiltonian.*

Moreover, Moon [7] observed that every strong tournament of order at least 3 is vertex-3-pancyclic; our next theorem is an extension of Moon's result.

Theorem 4.3. *For $k = 2, 3, 4$ every strong k-traceable oriented graph of order at least $k+1$ is vertex-$(k+1)$-pancyclic.*

Proof. Consider first the case $k = 4$. Let D be a strong 4-traceable oriented graph of order at least 5. First we show that every vertex in D that is contained in a 3-cycle also is contained in a 5-cycle. Let $C = v_1v_2v_3v_1$ be a 3-cycle in D. Let $U = N^-_{D-V(C)}(C)$ and $W = N^+_{D-V(C)}(C)$. Since D is strong and $n(D) \geq 5$, $U \neq \emptyset$ and $W \neq \emptyset$, and since D is 4-traceable, $U \cup W = V(D) - V(C)$. Suppose r and s are two distinct vertices in $U \cap W$. Then $\langle V(C) \cup \{r, s\}\rangle$ is a strong 4-traceable oriented graph of order 5 and hence contains a 5-cycle by Theorem 4.2. We therefore assume $|U \cap W| \leq 1$. W.l.o.g., we may assume that $W - U \neq \emptyset$. Since D is strong, there is a path from $W - U$ to C, passing through U; i.e., some vertex w in $W - U$ has an out-neighbour $u \in U$ and again $\langle V(C) \cup \{u, w\}\rangle$ contains a 5-cycle. It therefore follows from Lemma 4.1 that every vertex in D is contained in a 4- or a 5-cycle.

We now show that if a vertex x is contained in a t-cycle, for some $4 \leq t \leq n-1$, then x is also contained in a $(t+1)$-cycle; this will allow us to conclude that x is contained in cycles of all lengths $5, 6, \ldots, n$.

Now suppose $C = v_1v_2 \ldots v_tv_1$ is a t-cycle containing x, where $4 \leq t \leq n-1$. Let $U = N^-_{D-V(C)}(C)$ and $W = N^+_{D-V(C)}(C)$. Since D is strong, $U \neq \emptyset$ and $W \neq \emptyset$, and since D is 4-traceable, $U \cup W = V(D) - V(C)$. Suppose there is a vertex $v \in U \cap W$. Then $\langle V(C) \cup \{v\}\rangle$ is a strong 4-traceable oriented graph of order $t+1 \geq 5$ and hence, by Theorem 4.2 it contains a $(t+1)$-cycle. Thus we may assume that $U \cap W = \emptyset$.

Now suppose $w \in W$ and two consecutive vertices, say v_2 and v_3, are both in $N^0[w]$. Since D is 4-traceable, $|N^0[w]| \leq 3$. Then $V(C) - \{v_2, v_3\} \subseteq N^-(w)$. Since $\langle\{v_1, v_2, v_3, w\}\rangle$ is traceable, $v_3v_1 \in A(D)$. Since D is strong, w has an out-neighbour u in $D - V(C)$, and since $\langle\{v_2, v_3, u, w\}\rangle$ is traceable, $uv_2 \in A(D)$. Now $C' = v_t wuv_2v_3 \ldots v_t$ is a $(t+1)$-cycle (which contains x, unless $x = v_1$).

For each $i \in \{4, 5, \ldots, t-1\}$, $\langle\{v_1, v_{i-1}, v_i, w\}\rangle$ is traceable, so if $v_{i-1}v_1 \in A(D)$, then v_i must be adjacent with v_1; but if $v_1v_i \in A(D)$, then $C^* = v_{t-1}wuv_2v_3 \ldots v_{i-1}v_1v_iv_{i+1} \ldots v_{t-1}$ is a $(t+1)$-cycle and either C' or C^* contains x. We may therefore assume that if $v_{i-1}v_1 \in A(D)$, then $v_iv_1 \in A(D)$ (for each $i \in \{4, 5, \ldots, t-1\}$). Since, indeed, $v_3v_1 \in A(D)$, it follows that every vertex in $\{v_3, v_4, \ldots, v_{t-1}\}$ is an in-neighbour of v_1. Thus $C^* = v_{t-1}v_1wuv_2v_3 \ldots v_{t-1}$ is a $(t+1)$-cycle and either C' or C^* contains x. Hence no vertex in W (and similarly in U) has two consecutive nonneighbours on C.

Let $u \in U$ and suppose some vertex, say v_2, on C is not a neighbour of u. Then $v_1, v_3 \in N^+(u)$. We note that, since $\langle\{u, v_2, v_3, v_t\}\rangle$ is traceable, $v_tv_2 \in A(D)$, and since $\langle\{u, v_1, v_2, v_t\}\rangle$ is traceable, $uv_t \in A(D)$. Since D is strong, u has an in-neighbour $w \in W$.

Suppose first $v_4 \notin N^+(u)$. Then, since $\langle\{u, v_2, v_3, v_4\}\rangle$ and $\langle\{u, v_1, v_2, v_4\}\rangle$ are traceable, $v_4v_2, v_1v_4 \in A(D)$. Thus $v_2w \in A(D)$, because $\langle\{u, w, v_2, v_4\}\rangle$ is traceable. But then x belongs to one of the $(t+1)$-cycles $v_2wuv_3v_4 \ldots v_tv_2$ and $v_tv_2wuv_1v_4v_5 \ldots v_t$.

So suppose $uv_4 \in A(D)$. If $v_2w \in A(D)$, then x belongs to one of the $(t+1)$-cycles $v_1v_2wuv_4v_5 \ldots v_tv_1$ and $v_2wuv_3v_4 \ldots v_tv_2$. If $v_2w \notin A(D)$, then, since $\langle\{u, w, v_2, v_3\}\rangle$ is traceable, $v_3w \in A(D)$ and so $C' = v_2v_3wuv_4v_5 \ldots v_tv_2$ is a $(t+1)$-cycle. Now if $t = 4$, then x belongs to either C' or the $(t+1)$-cycle $v_1v_2v_3wuv_1$ and if $t \geq 5$, then, since each of $\langle\{u, v_2, v_3, v_5\}\rangle$ and $\langle\{u, w, v_2, v_5\}\rangle$ is traceable, $uv_5 \in A(D)$, and thus x belongs to either C' or the $(t+1)$-cycle $v_1v_2v_3wuv_5v_6 \ldots v_tv_1$.

We may therefore assume that every vertex on C is adjacent with every vertex in $U \cup W$. Thus $C' = v_twuv_2v_3 \ldots v_t$ and $C^* = v_1wuv_3v_4 \ldots v_tv_1$ are $(t+1)$-cycles and either C' or C^* contains x. This proves the case $k = 4$.

Now suppose D is a strong 3-traceable oriented graph of order at least 4. If $n = 4$, the result follows from Theorem 4.2, so we suppose $n \geq 5$. As observed in [11], every 3-traceable oriented graph is also 4-traceable, so it follows from the above that D is vertex 5-pancyclic. Therefore, it remains only to show that every vertex x of D belongs to a 4-cycle.

By Lemma 4.1, we may assume x is on a 3-cycle $C = v_1v_2v_3v_1$. Let $U = N^-_{D-V(C)}(C)$ and $W = N^+_{D-V(C)}(C)$. Since D is strong and $n(D) \geq 5$, $U \neq \emptyset$ and $W \neq \emptyset$, and since D is 3-traceable, $U \cup W = V(D) - V(C)$. Suppose $v \in U \cap W$. Then $\langle V(C) \cup \{v\}\rangle$ is a strong 3-traceable oriented graph of order 4 and hence has a 4-cycle by Theorem 4.2. We therefore assume $|U \cap V| = \emptyset$ and hence there exist $w \in W$ and $u \in U$ such that $wu \in A(D)$. Since D is 3-traceable, it is easy to see that every vertex of C is an in-neighbour of w and an out-neighbour of u. Hence x belongs to a 4-cycle. This proves the claim for $k = 3$.

The case $k = 2$ (Moon's result) is a direct consequence of Lemma 4.1 and the fact that every 2-traceable oriented graph is 3-traceable. $\square$

It is straightforward to see that 3-traceable oriented graphs are local tournaments; furthermore, Theorem 4.3 implies that every strong 3-traceable oriented graph of order at least 4 is vertex 4-pancyclic. This property is not shared by local tournaments in general; consider, for example, $\overrightarrow{C}_n$, $n \geq 5$.

It is shown in [12] that, for every $k \geq 5$, there exist nonhamiltonian strong k-traceable oriented graphs of order n for every $n \geq k$. Thus Theorems 4.2 and 4.3 do not extend beyond $k = 4$. However, Theorem 4.3 implies that for $k = 2, 3, 4$, every strong k-traceable oriented graph with girth at least k is girth-pancyclic, and this result can be extended; in fact, we shall prove that it holds for every $k \geq 2$. The proof relies on the following result, which generalizes (for strong oriented graphs) the fact that every 3-traceable oriented graph is a local tournament.

Lemma 4.4. *If $k \geq 2$ and D is a strong k-traceable oriented graph with girth $g \geq k$, then D is a local tournament.*

Proof. Let $x \in V(D)$ and suppose $\{u, w\} \subseteq N^-(x)$ such that u and w are nonadjacent. Then $k \geq 4$, since $\langle\{x, u, w\}\rangle$ is nontraceable. Since D is strong, there exists an $x - u$ path as well as an $x - w$ path in D. We may assume that the $x - u$ path does not contain w. Since $g \geq k$, such a path P has order at least k; say $P = xv_1v_2 \ldots v_tu$, where $t \geq k-2$. Now consider $S = \{u, w, x, v_1, v_2, \ldots, v_{k-3}\}$. Then $|S| = k$, but since $g \geq k$, it follows that neither u nor w has any in-neighbour in $\langle S\rangle$, and hence $\langle S\rangle$ is nontraceable, contradicting that D is k-traceable. This proves that $\langle N^-(x)\rangle$, and similarly $\langle N^+(x)\rangle$, is a tournament. $\square$

Lemma 4.4 now enables us to prove the following result, which will also be used in Section 5.

Lemma 4.5. *Let $k \geq 3$ and suppose D is a strong k-traceable oriented graph with girth $g \geq \max\{4, k\}$. If C is a cycle in D and x is any vertex in $V(D) - V(C)$, then x has an out-neighbour on C whose predecessor is an in-neighbour of x on C.*

Proof. Let $C = v_1v_2 \ldots v_tv_1$ be a cycle of order t in D and suppose $x \in V(D) - V(C)$. Since D is k-traceable and $t \geq g \geq k$, x has a neighbour on C, say v_1.

First, suppose $v_1 \in N^+(x)$ and that no predecessor of an out-neighbour of x on C is an in-neighbour of x. By Lemma 4.4 and the fact that $\{v_t, x\} \subseteq N^-(v_1)$, $v_t \in N^+(x)$. By repeating the argument, we find that every vertex on C is an out-neighbour of x. Since C has two nonadjacent vertices, this implies that $\langle N^+(x)\rangle$ is not a tournament, contradicting Lemma 4.4. Thus the result is proved in this case.

If $v_1 \in N^-(x)$, then a similar argument shows that some successor of an in-neighbour of x is an out-neighbour of x, and again the result is proved. $\square$

Theorem 4.3 and Lemma 4.5 imply that for every $k \geq 2$, a strong k-traceable oriented graph D with girth at least k has a cycle of every length from $g(D)$ to $n(D)$. We have thus proved the following.

Theorem 4.6. *If $k \geq 2$ and D is a strong k-traceable oriented graph with girth at least k, then D is girth-pancyclic.*

In Theorem 4.6 "girth-pancyclic" cannot be strengthened to "vertex-girth-pancyclic", since, for example, the $(k+1)$-cycle $v_1v_2\dots v_{k+1}v_1$ plus the chord v_1v_3 is a strong k-traceable oriented graph with girth k but has a vertex that does not lie on a k-cycle. Furthermore, the girth condition in Theorem 4.6 cannot be relaxed, since adding the chord v_1v_4 to the cycle $v_1v_2\dots v_{k+1}v_1$ yields a strong k-traceable oriented graph of order $k+1$ and girth $k-1$ that does not contain a k-cycle.

Thomassen [9] showed that the Cartesian products $\overrightarrow{C}_p \times \overrightarrow{C}_{mp-1}$ (where $p \geq 3, m \geq 1, mp \geq 4$) are hypohamiltonian oriented graphs of order $n = p(mp-1)$—i.e., they are nonhamiltonian but with each vertex-deleted subdigraph being hamiltonian. It follows that these oriented graphs are $(n-1)$- and $(n-2)$-traceable. As was shown by Penn and Witte [8], the Cartesian product $\overrightarrow{C}_p \times \overrightarrow{C}_q$ contains a t-cycle if and only if $t = ap + bq$ for some pair a, b of relatively prime natural numbers. Thus we observe that $\overrightarrow{C}_p \times \overrightarrow{C}_{mp-1}$ ($p \geq 3, m \geq 1, mp \geq 4$) has cycles of lengths p, $mp-1$, and $p+mp-1$, but none of order t for any $\max\{p, mp-1\} < t < p+mp-1$. So the oriented graphs $\overrightarrow{C}_p \times \overrightarrow{C}_{mp-1}$ ($p \geq 3, m \geq 1, mp \geq 4$) are neither weakly pancyclic nor hamiltonian. In particular, they provide examples of strong k-traceable oriented graphs of order $k+1$ that have no $(k+1)$-cycle, as well as examples of strong k-traceable oriented graphs of order $k+2$ that do have a $(k+1)$-cycle but are not hamiltonian.

It remains an open question whether every strong k-traceable oriented graph of order at least $k+2$ has a cycle of every length from $k+1$ up to its circumference.

5. The order of k-traceable oriented graphs with a triangle-free strong component

For each $k \geq 2$ there exist infinitely many k-traceable oriented graphs. In fact, for each $k \geq 5$ there even exist infinitely many *nonhamiltonian strong* k-traceable oriented graphs (with girth 3) as shown in [12]. In this section we show that the situation changes significantly if we forbid 3-cycles. First, we establish an upper bound on the order of all strong k-traceable oriented graphs with girth $g \geq 4$. Then we establish an even smaller upper bound on the order of k-traceable oriented graphs that are not strong but have a strong component X with $g(X) \geq 4$.

The following obvious observations concerning girth cycles will be used frequently.

Observation 5.1. *Let D be an oriented graph with girth $g \geq 4$ and let $C = v_1v_2v_3\dots v_gv_1$ be a girth cycle in D. Then the following hold.*

1. *C has no chords;*
2. *$N^-(v_1) \cap N^+(v_i) = \emptyset$ for $i = 2, 3, \dots, g-2$;*
3. *$N^+(v_1) \cap N^-(v_i) = \emptyset$ for $i = 4, 5, \dots, g$;*
4. *If $x \in V(D) - V(C)$ and x has both an in-neighbour and an out-neighbour on C, then $N_C(x)$ is contained in a set of three consecutive vertices of C with in-neighbour(s) preceding out-neighbour(s).*

We now consider *strong* triangle-free k-traceable oriented graphs.

Theorem 5.2. *Let $k \geq 3$ and let D be a strong k-traceable oriented graph of order n and girth $g \geq 4$. Then the following hold.*

1. *$g \leq k+1$;*
2. *If $g = k+1$, then $n = k+1$;*
3. *If $g = k$, then $n \leq 7$ if $k = 4$, and $n \leq k+1$ if $k \geq 5$;*
4. *If $g \leq k-1$, then*

$$n(D) \leq \begin{cases} 5k-11 & \text{if } g = 4 \\ 3k-8 & \text{if } g = 5 \\ 2k-g+2 & \text{if } g \geq 6 \text{ and } k-g \text{ is odd} \\ 2k-g+1 & \text{if } g \geq 6 \text{ and } k-g \text{ is even.} \end{cases}$$

Proof. Let $C = v_1v_2\dots v_gv_1$ be a girth cycle in D and let $X = V(D) - V(C)$.

1. This follows from Lemma 4.1.
2. Let $g = k+1$. Suppose $n(D) > k+1$ and let $x \in X$. By Lemma 4.5 and Observation 5.1(4) we may assume w.l.o.g. that $N_C(x) \subseteq \{v_1, v_2, v_3\}$. This together with Observation 5.1(1) implies that $\langle\{x, v_2, v_4, \dots, v_{k+1}\}\rangle$ is nontraceable, contradicting the k-traceability of D. Hence $n(D) = k+1$.

3. First, let $g = k = 4$. Suppose $n \geq 8$. Let $X' = \{x_1, x_2, x_3, x_4\} \subseteq X$. By Lemma 4.5 and Observation 5.1, every vertex in X' has at most three neighbours in C. But $\langle V(C) \cup X' \rangle$ is a 4-traceable oriented graph of order 8, so by Lemma 3.1 it has minimum degree at least 5, which implies that every vertex in C has at least three neighbours in X'. Hence $|N_C(x_i)| = 3$ for $i = 1, 2, 3, 4$. Assume without loss of generality that v_i is the nonneighbour of x_i on C, $i = 1, 2, 3, 4$. Then, by Observation 5.1, $v_2 \in N^-(x_1)$ and $v_4 \in N^+(x_1)$. Similarly, $v_4 \in N^-(x_3)$ and $v_2 \in N^+(x_3)$. Thus $x_1v_4x_3v_2x_1$ is a 4-cycle and hence has no chords, which implies that x_1 and x_3 are nonadjacent. Let $H = \langle\{x_1, v_1, v_3, x_3\}\rangle$. Then the complement of the underlying graph of H is the 4-cycle $x_1v_1v_3x_3x_1$, so H cannot contain a path of order 4, contradicting our assumption that D is 4-traceable. So $n \leq 7$.

   Now let $g = k = 5$. Suppose $n \geq 7$ and let x_1, x_2 be any two distinct vertices in X. The subdigraph $\langle V(C) \cup \{x_1, x_2\}\rangle$ is 5-traceable with order 7 and hence has minimum degree at least 3. Thus every vertex on C has at least one neighbour in $\{x_1, x_2\}$. By Lemma 4.5 and Observation 5.1, we may assume $N_C(x_1) = \{v_1, v_2, v_3\}$ with $v_1 \in N^-(x_1)$, $v_3 \in N^+(x_1)$, and $v_4, v_5 \in N(x_2)$. Then, by Observation 5.1, $v_2 \notin N(x_2)$. By considering the girth cycle obtained from C by replacing v_2 with x_1 we note that also $x_1 \notin N(x_2)$. Thus neither x_1 nor v_2 has any neighbours in $\{x_2, v_4, v_5\}$, so $\langle\{x_1, x_2, v_2, v_4, v_5\}\rangle$ is nontraceable, a contradiction. So $n \leq 6$.

   Now let $g = k \geq 6$. Suppose some vertex on C, say v_1, has degree at least 4. Then $d_X(v_1) \geq 2$. Now let S consist of $V(C) - \{v_3, v_{g-1}\}$ together with two vertices in $N_X(v_1)$. Since $g > 3$, there is no arc from $N^+(x)$ to $N^-(x)$ for any $x \in V(D)$. Hence it follows from Observation 5.1 that in $\langle S\rangle$ there is no path from v_4 to v_1 and also no path from v_1 to v_4, so $\langle S\rangle$ is nontraceable. This proves that $d(v_i) \leq 3$ for $i = 1, \ldots, k$. Hence, by Lemma 3.1, $n \leq k + 2$. But, $\{v_1, v_3, v_5\}$ is an independent set, since $g \geq 6$. Now it follows from Corollary 3.5 that n and k cannot have the same parity; in particular, $n \neq k + 2$. So $n \leq k + 1$.
4. Let $g \leq k - 1$.

   First, let $g = 4$. Suppose $d^-(v_1) \geq 2k - 4$. Then $d_X^-(v_1) \geq 2k - 5$. By Observation 5.1, v_2 has no out-neighbour in $N^-(v_1)$, and since D is k-traceable, v_2 has at most $k - 2$ nonneighbours in D (not including v_2), hence at most $k - 3$ nonneighbours in $N_X^-(v_1)$. Thus $|N_X^-(v_1) \cap N_X^-(v_2)| \geq 2k - 5 - (k - 3) = k - 2$. Now let S consist of v_1, v_3, and $k - 2$ vertices in $N_X^-(v_1) \cap N_X^-(v_2)$. By Observation 5.1, v_3 has no out-neighbour in $N^-(v_2)$. Hence neither v_1 nor v_3 has any out-neighbours in S, so $\langle S\rangle$ is nontraceable, contradicting the k-traceability of D. This proves that $d^-(v_1) \leq 2k - 5$ and it follows by a symmetric argument that $d^+(v_1) \leq 2k - 5$. Hence $d(v_1) \leq 4k - 10$, so Lemma 3.1 implies that $n - k + 1 \leq 4k - 10$, i.e., $n \leq 5k - 11$.

   Next, let $g = 5$. Suppose $d^-(v_1) \geq k - 2$. By Observation 5.1, v_3 has no out-neighbour in $N^-(v_1)$, so if S consists of v_1, v_3 and $k - 2$ vertices in $N^-(v_1)$, then $\langle S\rangle$ is nontraceable. This proves that $d^-(v_1) \leq k - 3$. Similarly, $d^+(v_1) \leq k - 3$, so $d(v_1) \leq 2k - 6$. Now Lemma 3.1 implies $n - k + 1 \leq 2k - 6$, i.e., $n \leq 3k - 7$.

   Now suppose $n = 3k - 7$. Then $n - k + 1 = 2k - 6$ and it follows from the above that $d^-(v_i) = d^+(v_i) = k - 3 = (n - k + 1)/2$ for $i = 1, 2, \ldots, 5$. Hence Lemma 3.4 implies that $N^+(v_1) \cap N^+(v_3) = \emptyset$. But, by Observation 5.1, $N^-(v_1) \cap N^+(v_3) = \emptyset$. Hence $N(v_1) \cap N^+(v_3) = \emptyset$. Let $Y = X - N_X(v_1)$. Then $|Y| = 3k - 7 - 5 - (2k - 8) = k - 4 = d_X^+(v_3)$. This implies that $N_X^+(v_3) = Y$. But, by Lemma 3.4, $N_X^-(v_3) \cap N_X^-(v_1) = \emptyset$, so $N_X^-(v_3) = N_X^+(v_1)$. Now, if w is any vertex in $N^+(v_1)$, then $v_1wv_3v_4v_5v_1$ is a girth cycle of D and hence has no chords. This implies that v_5 has no neighbours in $N^+(v_1)$. Again, by Lemma 3.4, $N^+(v_5) \cap N^+(v_3) = \emptyset$, so $N_X^+(v_5) = N_X^-(v_1)$ and $N_X^-(v_5) = Y$. By a symmetric argument, $N_X^-(v_4) = Y = N_X^+(v_2)$. Now let $y \in Y$. Then $v_1v_2yv_5v_1$ is a cycle of order 4, contradicting that $g = 5$. Hence $n \leq 3k - 8$.

   Now let $g \geq 6$. Suppose $d(v_1) \geq k - g + 4$. Then $d_X(v_1) \geq k - g + 2$. Now let S consist of $V(C) - \{v_3, v_{g-1}\}$ together with $k - g + 2$ vertices in $N_X(v_1)$. Then it follows from Observation 5.1 that in $\langle S\rangle$ there is no path from v_4 to v_1 and also no path from v_1 to v_4, so $\langle S\rangle$ is nontraceable. This proves that $d(v_1) \leq k - g + 3$ and hence it follows from Lemma 3.1 that $n \leq 2k - g + 2$.

   Now assume $k - g$ is even and $n = 2k - g + 2$. Then n and k have the same parity and it follows from Lemma 3.1 and the above that $n - k + 1 \leq d(v_i) \leq k - g + 3 = n - k + 1$, i.e., $d(v_i) = n - k + 1$ for $i = 1, 2, \ldots, g$. But $\{v_1, v_3, v_5\}$ is an independent set, since $g \geq 6$, so we have a contradiction to Corollary 3.5. Hence in this case $n \leq 2k - g + 1$. □

Remark 5.3. $\overrightarrow{C}^2_{11}$ realizes the bound on n given in Theorem 5.2 in the case $g = 6$, $k = 8$, and $\overrightarrow{C}^2_7$ realizes the bound in the case $k = g = 4$.

In particular, Theorem 5.2(2) implies that for every $k \geq 3$ the only strong k-traceable oriented graph with girth $k + 1$ is $\overrightarrow{C}_{k+1}$.

Also, if D is a strong k-traceable oriented graph with girth $k \geq 5$, then it follows from Theorem 5.2(3) that $n(D) \leq k + 1$; hence in these cases Lemma 4.5 implies that D is either a $\overrightarrow{C}_k$ or a $\overrightarrow{C}_{k+1}$ *plus one short forward chord*, or a $\overrightarrow{C}_{k+1}$ *plus two consecutive short forward chords*.

Using Lemmas 3.1 and 4.4, Theorems 4.2, 5.2(2) and (3), and Observation 5.1, one can see by inspection that the only strong 4-traceable oriented graphs with girth at least 4 are the ones depicted in Fig. 3.

For k-traceable oriented graphs that are *not* strong, the following result is useful.

Lemma 5.4. *Suppose D is a k-traceable oriented graph and X is a strong component of D. If t is an integer such that $2 \leq t \leq \min\{n(X), k-1\}$ and X is not t-traceable, then $n(D) - n(X) \leq k - t - 1$.*

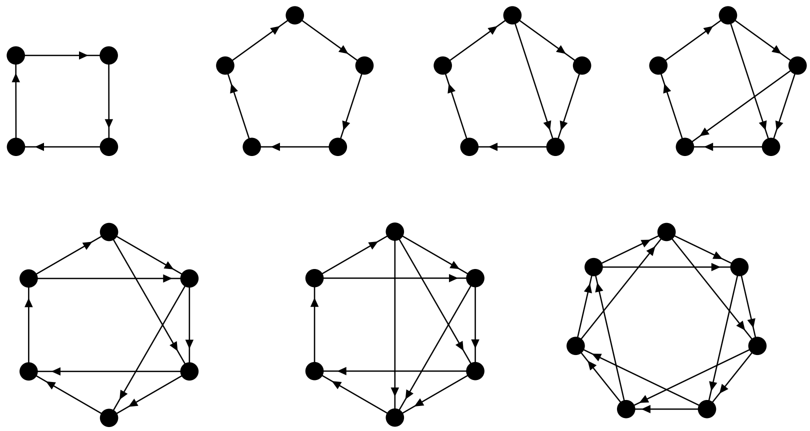

Fig. 3. The strong 4-traceable oriented graphs with girth at least 4.

Proof. Suppose, to the contrary, that $n(D) - n(X) \geq k - t$. By our assumption that X is not t-traceable, X contains a nontraceable induced subdigraph H with $n(H) = t$. Now let H^* be the subdigraph of D induced by $V(H)$ together with $k - t$ vertices of $D - V(X)$. Then, since $n(H^*) = k$ and D is k-traceable, H^* has a hamiltonian path P. But then the intersection of P with X is a hamiltonian path of H, contradicting that H is nontraceable. □

We now use Theorem 5.2 together with Lemma 5.4 to prove the following result for oriented graphs that are *not strong*.

Theorem 5.5. *Let $k \geq 3$ and let D be a k-traceable oriented graph that is not strong. If D has a strong component X such that $g(X) \geq 4$, the following hold.*

1. *$g(X) \leq k$ (and hence $k \geq 4$);*
2. *If $g(X) = k$, then $n(D) = k + 1$;*
3. *If $g(X) \leq k - 1$, then*

$$n(D) \leq \begin{cases} 5k - 15 & \textit{if } g(X) = 4 \\ 3k - 10 & \textit{if } g(X) = 5 \\ 2k - g(X) + 1 & \textit{if } g \geq 6. \end{cases}$$

Proof. Let

$$r = n(D) - n(X).$$

Let C be a girth cycle of X and let u, v be two nonconsecutive vertices of C. Then $C - \{u, v\}$ is a nontraceable subdigraph of X of order $g(X) - 2$. Thus X is not $(g(X) - 2)$-traceable. But $2 \leq g(X) - 2 \leq k - 1$, so it follows from Lemma 5.4 (taking $t = g(X) - 2$) that

$$r \leq k - g(X) + 1.$$

1. Our assumption that D is not strong implies that $r \geq 1$ and hence $k \geq g(X) \geq 4$.
2. If $g(X) = k$, then $r = 1$. But by Lemma 5.4, X is $(k - 1)$-traceable, so Theorem 5.2(2) implies that $n(X) = k$ and hence $n(D) = k + 1$.
3. Suppose $g(X) \leq k - 1$. Since $k - r \geq g(X) - 1 \geq 3$ and $n(X) = n(D) - r \geq k - r$, it follows from Lemma 5.4 that X is $(k - r)$-traceable.

   We now apply Theorem 5.2 to the strong $(k - r)$-traceable oriented graph X.

   If $g(X) = k - r + 1$, then, by Theorem 5.2(2), $n(X) = k - r + 1$, which implies that $n(D) = k + 1$ and so $n(D)$ satisfies the inequality stated in (3). Hence, for the remainder of the proof we may assume $g(X) \leq k - r$.

   Now suppose $g(X) = 4$. If $k - r = 4$, then, by Theorem 5.2(3), $n(X) \leq 7$. But $n(D) = n(X) + r$, so

   $$n(D) \leq 7 + r = 7 + (k - 4) = k + 3 < 5k - 15,$$

   since $k = r + 4 \geq 5$.

   If $k - r \geq 5$, then Theorem 5.2(4) implies that

   $$n(X) \leq 5(k - r) - 11$$

   and hence

   $$n(D) \leq 5k - 4r - 11 \leq 5k - 15.$$

Now suppose $g(X) = 5$. If $k - r = 5$, then, by Theorem 5.2(3), $n(X) \leq 6$, which implies

$$n(D) \leq 6 + r \leq k + 1 < 3k - 10,$$

since $k = r + 5 \geq 6$.

If $k - r \geq 6$, then Theorem 5.2(4) implies that

$$n(X) \leq 3(k - r) - 8$$

and hence

$$n(D) \leq 3k - 2r - 8 \leq 3k - 10.$$

Finally, suppose $g(X) \geq 6$. If $k - r = g(X)$, then, by Theorem 5.2(3), $n(X) \leq k - r + 1$, which implies

$$n(D) \leq k - r + 1 + r = k + 1 < 2k - g(X) + 1,$$

since $g(X) < k$.

If $k - r \geq g(X) + 1$, then Theorem 5.2(4) implies that

$$n(X) \leq 2(k - r) - g(X) + 2$$

and hence

$$n(D) \leq 2k - r - g(X) + 2 \leq 2k - g(X) + 1. \quad \square$$

In particular, Theorems 5.2 and 5.5 imply the following.

Corollary 5.6. *If D is a k-traceable oriented graph with a strong component X such that $g(X) \geq 6$, then $n(D) \leq 2k - 4$.*

Proof. If D is strong, then $D = X$ and so $k \geq 5$ by Theorem 5.2(1); if D is not strong, then $k \geq 6$ by Theorem 5.5(1). In either case, these two theorems imply that $n(D) \leq \max\{k + 1, 2k - 4\} = 2k - 4$, since $k \geq 5$. $\square$

6. The Traceability Conjecture and the Path Partition Conjecture for oriented graphs

An (a, b)*-partition* of a digraph D is a partition of $V(D)$ such that $\lambda(\langle A\rangle) \leq a$ and $\lambda(\langle B\rangle) \leq b$. If D has an (a, b)-partition for every pair of positive integers (a, b) such that $a + b = \lambda(D)$, then we say D is λ*-partitionable*.

The Path Partition Conjecture (for digraphs) may be stated as follows.

Conjecture 1 (*PPC*)**.** *Every digraph is λ-partitionable.*

The PPC for *un*directed graphs (which is equivalent to the PPC for symmetric digraphs) was stated in 1982 by Laborde et al. in [6]. The PPC has been shown to hold for special classes of graphs and digraphs (see e.g. [10,3,4]) but the conjecture seems very difficult to settle in general.

Our interest in k-traceable oriented graphs arose from our study of the PPC *for 1-deficient oriented graphs*. This special case of the PPC is an intriguing conjecture in its own right (see [12,13]). As shown in [12], the PPC will hold for 1-deficient oriented graphs if the following conjecture, known as the Traceability Conjecture, is true.

Conjecture 2 (*TC*)**.** *For $k \geq 2$, every k-traceable oriented graph of order at least $2k - 1$ is traceable.*

As mentioned earlier, for $k = 2, 3, 4, 5, 6$, *every* k-traceable oriented graph is traceable. Furthermore, it is shown in [11] that for each $k \geq 2$ there exists an integer $t(k)$ $(< 2k^2)$ such that every k-traceable oriented graph of order at least $t(k)$ is traceable. We have not yet succeeded in reducing the upper bound on $t(k)$ to a function that is linear in k when $k \geq 7$.

Theorem 5.5 resulted from our (as yet unsuccessful) attempts at proving the Traceability Conjecture for triangle-free oriented graphs. It is shown in [12] that if D is a nontraceable k-traceable oriented graph of order at least $2k - 1$, then D contains a nontrivial nonhamiltonian strong component and all its other strong components are tournaments. Thus, if one could decrease the upper bound on n in Theorem 5.2 by adding the requirement that D be *nonhamiltonian*, this would lead to a decrease in the upper bound on n in Theorem 5.5 under the added assumption that D be *nontraceable*. Such a result could bring us a step closer to proving the TC for oriented graphs that have a triangle-free strong component.

Corollary 5.6 implies that, in order to prove the TC, we only need to consider k-traceable oriented graphs whose strong components all have girth at most 5. We now show that Corollary 5.6 also implies that the PPC holds for 1-deficient oriented graphs that have a strong component with girth at least 6.

Theorem 6.1. *Suppose D is a 1-deficient oriented graph that has a strong component X with $g(X) \geq 6$. Then D is λ-partitionable.*

Proof. Suppose (a, b) is a pair of positive integers such that $a + b = \lambda(D)$. Assume $a \leq b$. Since D is 1-deficient, $n(D) = \lambda(D) + 1 = a + b + 1 \geq 2(a + 1) - 1$. Hence, by Corollary 5.6, D is not $(a + 1)$-traceable. Therefore, D has a nontraceable induced subdigraph H of order $a + 1$. Let $A = V(H)$ and $B = V(D) - V(H)$. Then (A, B) is an (a, b)-partition of D. $\square$

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